

# Phonon modes in the Frenkel-Kontorova chain: exponential localization and the number theory properties of frequency bands

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We study numerically phonon modes of the classical one-dimensional Frenkel-Kontorova chain, in the regime of pinned phase characterized by the phonon gap and devil's staircase, as well as by a large number of states (configurational excitations), which energy splitting from the ground state is exponentially small. We demonstrate, these states behave like disorder media: their phonon modes are *exponentially* localized, in contrast to the phonon modes in the ground state, where phonons are *prelocalized* only.

We demonstrate also, the phonon frequency spectrum of the ground state has an hierarchical structure, a direct manifestation of hierarchical spatial structure, found for the ground state of the FK chain in our recent work.

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## I. INTRODUCTION

The most trivial disorder originates in media due to random static impurities (see, e.g. [1]). However, another but very interesting possibilities are glasses, which have a huge number of (meta)stable degenerated states. Originally glassy system has a homogeneous Hamiltonian with no intrinsic random parameter, and disorder occurs in it dynamically. Recently [2] we have demonstrated that a popular Frenkel-Kontorova model [3] presents an example of glassy system, which has a lot of static states, known as *configurational excitations* of the classical ground state, with energy splitting extremely (exponentially) small. As it was shown in [4], this model has a nontrivial quantum dynamics, the quantum phase transition: if quantum parameter exceeds some critical value, the "pinned" glassy phase turns into "sliding" phonon gas.

The Frenkel-Kontorova model (FK) [3] is widely used [5, 6, 7, 8, 9, 10, 11, 12, 13, 14] in the solid state physics to get insight on generic properties of noncommensurate systems. Its ground state, which is rather quasiperiodical [2, 9, 15] than periodical, attracts also an attention [9, 10, 16, 17, 18] as some interplay [17] between order and disorder [19].

This model describes a chain of atoms/particles inter-

acting with elastic forces, placed in periodic potential, which period differs from a mean interparticle distance. The ground state (GS) of this model is defined as a static, equilibrium configuration of the chain, that corresponds to the *absolute* minimum of the chain potential energy. The ground state is unique and has some special order of particles in the chain, that was discovered by Aubry [9, 15] more than twenty years ago. The positions of atoms in the chain are described by an area preserving map, which is well known in the field of dynamical chaos as the Chirikov standard map [20]. The ratio of the mean interparticle distance to a period of the external potential in the FK model determines the rotation number of the invariant curves of the map, while the amplitude of the periodic potential gives the value of the dimensionless parameter  $K$ . For  $K < K_c$  the KAM curves are smooth and the spectrum of longwave phonon excitations in the chain is characterized by a linear dispersion law starting from zero frequency. In this regime the chain can freely slide along the external field (the "sliding" phase). On the contrary, for  $K > K_c$  the KAM curves are destroyed and replaced by an invariant Cantor set, which is called cantorus. In this regime the phonon spectrum has a gap, and the chain is pinned ("pinned" phase). Later, on the example of Ising spin model to which the FK model can be approximately reduced [21] it has been shown [22] that the GS has some well defined hierarchical structure, which particular details are determined by number properties of the ratio of the mean interparticle distance to the period of the external field. Recently our numerical study [2] of the original FK model in the pinned phase has

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shown, that the GS has indeed an hierarchical structure, but in some important details different from predicted in [22].

In short, we put our attention to the striking fact, that in the pinned phase of the FK chain there are some particles, which positions exponentially close to bottoms of wells of the external potential: corresponding external force acting to such particle is extremely close to zero. Obviously, in static equilibrium each of these particles can be considered as some dummy “glue” that only couples two adjacent parts of the chain, which ends has almost identical (exponentially close) tension forces.

Another important observation is that small deviations of glue particles from their well bottoms are *grouped* into well defined hierarchically ordered scales. Now, if one cut the chain into fragments via glue particles, which belong to level with least deviation from well bottoms, one gets several fragments of *two* sizes, or two species of some “bricks” [30]. Then one may repeat the procedure, cutting “bricks” via “glue” particles that belong to next scale of deviations and getting two new species of smaller bricks, and so far. At the last step one gets two species of smallest possible bricks with *no* glue particle inside: smaller bricks  $A^{(0)}$  which consist of 2 particles inside a single well, and larger brick  $B^{(0)}$  which consist of 4 particles (two pairs in two adjacent wells). In this way one gets an hierarchically ordered set of brick species  $\{A^{(i)}, B^{(i)}\}$  with very simple composition rules [2]:

$$A^{(i+1)} = B^{(i)}gA^{(i)}gB^{(i)}, \quad (1)$$

$$B^{(i+1)} = B^{(i)}gA^{(i)}gB^{(i)}gA^{(i)}gB^{(i)}, \quad (2)$$

where symbol  $g$  denotes of an insertion of glue particle, which “glues” two adjacent bricks. The difference of tension forces at boundaries of bricks  $A^{(i)}$  and  $B^{(i)}$  is exponentially small and decrease rapidly with a number of the hierarchy level. In principle, these rules are sufficient to construct a GS for a FK chain of any length, if the rotation number parameter of the chain approximates the mean golden value  $\nu = (\sqrt{5} - 1)/2$ .

Besides the GS there exists “configurational excitation states” (CES), presented by static equilibrium configurations corresponding to *local* (rather than absolute) minima of the chain potential, with energy very close to GS. Within the picture just outlined above CES correspond to different permutations of bricks [2], therefore the number of them can be combinatorically huge. At any accessible small temperature their contributions can dominate over the contribution of GS.

In this paper we address to phonon excitations of the chain, small vibrations around static GS and CES configurations of the chain. These excitations are relevant as for heat transport properties [18, 23] of the chain, as for some quantum effects [24], especially in the quasiclassical limit. As in the previous paper, we concentrate on the case of pinned phase of the chain, which corresponds to a nonzero phonon gap. We start with analysis of the

structure of the phonon frequency spectrum in the GS. It is well known, that this spectrum is splitted into bands [10, 26] but, to our knowledge, up to now there is no clear explanation for origin of its splitting into particular bands and subbands. We show, that this splitting is a direct consequence of particular spatial structure of the chain in its GS. We have found, that this structure is also hierarchically ordered, with definite resemblance and distinctions with respect to a spatial structure of underlying GS.

Localisation properties of phonons in incommensurate one-dimensional chains are intensively studied in recent works [17, 18, 23, 25], with strong indications [23, 25], that phonon modes in the GS of FK chain are not localized, and even at edges of frequency bands they are rather *prelocalized*, than localized. We study also phonon properties as in GS, as in CES of the chain. Our results confirm, that in the GS phonon modes are only prelocalized. However, the situation appears to be quite different for CES. Even for CES, which energy splitting (in natural problem scale)  $\Delta U \leq 10^{-12}$  there are phonon modes, which are localized exponentially. Moreover, for CES with higher splitting  $\Delta U$  we see, that there are entire bands of exponentially localized phonon modes.

## II. THE MODEL.

The Hamiltonian of the FK model is

$$H = \sum_{i=1}^s \frac{P_i^2}{2} + \frac{(x_i - x_{i-1})^2}{2} - K \cos x_i. \quad (3)$$

The first term in the Hamiltonian is a kinetic energy, where we put masses of particles  $m = 1$ , the second term describes interparticle interaction with elasticity coefficient put to unity, while the third term corresponds to particle interaction with external periodical field with coupling constant  $K$ . All  $s$  particles are distributed over  $r$  period/wells of the external potential, which period, without any loss of generality is taken equal to  $2\pi$ . The ratio  $\nu = r/s$  gives [9] the rotational number of corresponding standard map [20].

We assume periodical boundary conditions:  $P_0 \equiv P_s$ ,  $x_0 \equiv x_s - L$ , where  $L = 2\pi r$  is the length of the chain. In our subsequent analysis we take (as some *typical* example of FK chain) the chain with  $r/s = 377/610$  as an approximation of the golden mean value  $\bar{\nu} = (\sqrt{5} - 1)/2$ , and parameter  $K = 2$  well above the critical value  $K_c(\nu = \bar{\nu}) = 0.971 \dots$  [27]. The technique to obtain GS and CES is described in our previous work [2]. Let us stress, that validity of the GS can be checked either by direct analysis [2] of its spatial structure, or proved by monotonous behavior [15] of its hall function. Note, that the GS is almost degenerate with enormous number of CES: there are hundreds of states with  $\Delta U = U_{ECS} - U_{GS} \leq 10^{-80}$ , while the number of ECS with  $\Delta U \leq 10^{-12}$  exceeds  $10^9$ !

Effective Hamiltonian for phonon modes can be obtained by expansion up to second order terms of the original FK chain Hamiltonian (3) around a chosen static configuration of the chain:

$$H^{ph} = \sum_{i=1}^s \frac{\Pi_i^2}{2} + \frac{1}{2} \sum_{i,k=1}^s \frac{\partial U}{\partial x_i \partial x_k} \psi_i \psi_k, \quad (4)$$

where  $\psi_i = x_i - \bar{x}_i$  are small deviations of particles from their equilibrium static positions  $\bar{x}_i$ , and  $\Pi_i$  are corresponding particle momenta. The elasticity matrix

$$R_{ik} \equiv \frac{\partial^2 U}{\partial x_i \partial x_k} = K \cos(\bar{x}_i) \delta_{ik} - \delta_{i,k+1} - \delta_{i+1,k}. \quad (5)$$

Solving the eigenvalue problem

$$(R - \omega^2 I) \psi = 0, \quad (6)$$

numerically, we get both the spectrum of phonon frequencies and corresponding vectors of phonon modes.

### III. PHONON FREQUENCY SPECTRUM

It is well known for a long time [10, 26], that frequency spectrum of phonons in the Frenkel-Kontorova chain is splitted into several bands. Main features of the spectrum for a chain in the GS look as very universal: (i) the number of main bands are independent of the parameter  $K$  as well as of a length of the chain; (ii) it was also noticed in [17] (without any explanation, either) that boundaries of the bands correspond to Fibonacci numbers; (iii) replacement in interparticle interactions elastic forces by the Lenard-Jones potential does not change qualitatively a pattern of splitting phonon spectrum into band [17].

Now we can explain these and more features as direct consequence of particular spatial structure [2] of the GS, which main relevant details are summarized in Introduction. The key point of our explanation is that the GS consists of large number of almost identical elements. For clarity sake, we consider a particular case of the FK chain with  $r/s = 377/610$ , but all our arguments can be easily generalized for a chain of different size.

Applying the picture of CS spatial structure outlined in Introduction to our particular case, we can see, that our CS can be presented as a composition of two bricks of 3d level of hierarchy:

$$610 = (g232g376), \quad (7)$$

with subsequent expansion, in accordance with composition rules (1),(2), into bricks of 2nd level:

$$232 = (88g54g88), \quad 376 = (88g54g88g54g88), \quad (8)$$

which, in turn, are expanded into bricks of 1st level:

$$54 = (20g12g20), \quad 88 = (20g12g20g12g20), \quad (9)$$

that, at last, are expanded into basic bricks of zero level:

$$12 = (4g2g4), \quad 20 = (4g2g4g2g4). \quad (10)$$

In this way it is easy to calculate, that the chain can be cut into pieces either as

$$144 \times g + 55 \times (2) + 89 \times (4), \quad (11)$$

or

$$34 \times g + 13 \times (12) + 21 \times (20), \quad (12)$$

or

$$8 \times g + 3 \times 54 + 5 \times 88, \quad (13)$$

or  $2 \times g + (232) + (376)$ , see (7). Mutual static disturbance of bricks is small and decreases exponentially with a level of hierarchy, therefore we have at any level of hierarchy a sequence in some order of two species of almost identical structures (bricks).

Now let us make some important remark. In our study in [2] of *static* structure of the chain, the glue particles play a some passive role only, which results in their specification as some dummy “glue”. However, in *dynamical* problem of motion particles in the chain, that we now address to, the role of glue particles becomes more important, since they have masses. Actually, in this case one should consider as repeating structures *elementary cells*, which we get if attach to each brick (e. g. at the left side) one glue particle[31].

At the basic level of hierarchy one have 55 elementary cells of 3 particles and 89 elementary cells of 5 particles, let them be called as cells  $\alpha^{(0)} = (g2)$  and  $\beta^{(0)} = (g4)$ , respectively. Each elementary cell, being isolated (and, e.g. periodically closed) has its own eigenfrequencies: three for a cell  $\alpha^{(0)}$  and five for a cell  $\beta^{(0)}$ . Then, if these cells in the chain be really isolated, then, see Fig.1 we get three bands each of 55 degenerated states (solid lines) which belongs to cells  $\alpha^{(0)}$ , and five bands each of 89 degenerated states (dotted lines).

Due to cells interactions all degenerated frequencies are really splitted into finite width bands, see Fig.1. The closer inspection of frequency spectrum shows, that seven largest breaks cut the exact spectrum in bands  $n = (1, 89), (90, 144), (145, 233), (234, 322), (323, 377), (378, 467), (468, 523), (524, 610)$  so the band widths have the same order 89, 55, 89, 89, 55, 89, 55, 89, exactly as the sequence of solid and dotted horizontal lines in Fig.1!

Now, let us remind, that the sequence of cells  $\alpha^{(0)}$  and  $\beta^{(0)}$  in our GS is not random but they belong to 34 cells of the next hierarchy level, see (12): 13 cells  $\alpha^{(1)} = (g12) = (g4g2g4)$  and 21 cells  $\beta^{(1)} = (g20) = (g4g2g4g2g4)$ . Again, if this cells would be decoupled, we could see bands, that consist of 13 and 21 degenerated states. Indeed, in the middle part of the Fig.1, where the central band is shown with greater resolution, we see a sequence of bands, that contain 21, 13, 21, (21+13), 21,

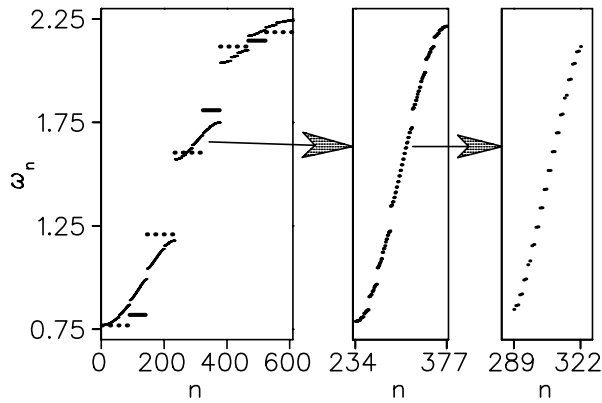


Figure 1: Frequency spectrum for FK model in the ground state. In the left box: the total spectrum of chain, in the middle and right boxes the central band and its central subband are shown with greater resolution. Solid and dashed horizontal lines show eigenfrequencies for isolated cells  $\alpha^{(0)}$  and  $\beta^{(0)}$  respectively.

13, 21 number of states, which is similar to that we have at the main level of hierarchy! [32] However, at higher resolution (the right box in Fig.1) the spectrum in central band becomes structureless.

This is not surprising, since the “cells” we have introduced are not weakly coupled objects with respect to phonon modes. In fact, what cells provides, is that some chain fragments have fixed periods; these fragments form locally their band structure. If bands of different fragments overlap, their levels are collectivised into one common band. The central band is the case, however in this case band levels are less sensitive to small variations introduced by extra regularities of next levels of hierarchy of spatial chain structure.

However, a quite new interesting phenomenon occurs, if one consider band, which belongs to a cell  $\beta^{(0)}$  but does *not* overlap with any band of the cell  $\alpha^{(0)}$ . Phonons with frequencies inside this band will be damped along the cells of the kind  $\alpha^{(0)}$ , these cells will play a role of some “potential barriers”, which decouple cells of the kind  $\beta^{(0)}$  each from other. Our chain contains 89 cells  $\beta^{(0)}$  and 55 cells  $\alpha^{(0)}$ , which is like to 89 particles separated by 55 barriers, or distributed among 55 wells. Now we have got a new effective Frenkel-Kontorova chain, where cells  $\beta^{(0)}$  play role some particles with potential barriers  $\alpha^{(0)}$  among them. For a clarity sake, let us take more graphical notations for these effective “barrier”  $\Lambda = \alpha^{(0)}$ , and “particle”  $\bullet = \beta^{(0)}$ . Then, secondary bricks, which occur in this effective FK chain, are  $\tilde{A} = \Lambda \bullet \bullet \Lambda$  and  $\tilde{B} = \Lambda \bullet \bullet \Lambda \bullet \bullet \Lambda$ , and corresponding elementary cells can be obtained adding to bricks one “glue” particle at the left:  $\tilde{\alpha} = \bullet \Lambda \bullet \bullet \Lambda$ ,  $\tilde{\beta} = \bullet \Lambda \bullet \bullet \Lambda \bullet \bullet \Lambda$ , having again three and five eigenfrequencies, respectively. Our effective chain has 8 cells  $\tilde{\alpha}$  and 13 cells  $\tilde{\beta}$ . Now, if in the frequency spectrum of new *effective* FK chain we take a band, which belongs to the cell  $\tilde{\beta}$  but not to the cell  $\tilde{\alpha}$ ,

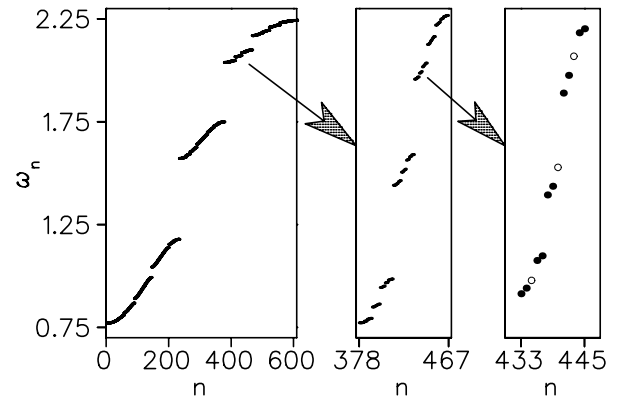


Figure 2: The same, as in Fig.1, but here a *fourth* band and its *fourth* subband are shown with greater resolution. In right figure box open and closed circles correspond to phonon modes of the cells  $\tilde{\alpha}$  and  $\tilde{\beta}$ , respectively.

it must contain 13 states.

We can repeat the whole procedure ones more; then we come to next level effective Frenkel-Kontorova chain with 13 effective particles  $\tilde{\beta}$  distributed among 8 potential barriers  $\tilde{\alpha}$ , and to next generation cells, one of the kind  $\tilde{\alpha}$  and two of the kind  $\tilde{\beta}$ . Numerical data presented in Fig.2 confirm our picture in all the details. Here we consider the fourth band of spectra, which is well resolved from other bands, see, the left box of Fig.2. It contains 89 states, the number of cells  $\beta^{(0)}$ . In the middle box we take again the fourth band which in turn contains 13 states, the number of cells  $\tilde{\beta}$ . At last, in the right box we show by open circles 3 states of the cell  $\tilde{\alpha}$  and by closed ones  $2 \times 5$  states of two cells  $\tilde{\beta}$ .

Note, that this new kind of hierarchy is quite different from that we found in [2] with respect to spatial structure, since the transformation rules between levels of hierarchy are more complicated.

In conclusion of this section it should be stressed, that all the universal features of the global band structure mentioned at the beginning are governed by nearest order in the chain. In contrast, the fine structure depends crucially on the far order. The latter is destroyed in the CES, therefore in CES the fine structure is washed out, and frequency spectra become smoother, see Fig.3. In particular, we see also in Fig.3 how two lowest bands are merged into common one.

#### IV. PHONON LOCALIZATION

The ordering of bricks, which persists in the GS, is gradually destroyed with configurational excitation of the chain. The lowest excitations are equivalent destruction of largest bricks due to permutations of bricks of preceeding level of hierarchy: the smaller bricks are destroyed, the higher is the excitation energy [2]. The number of almost

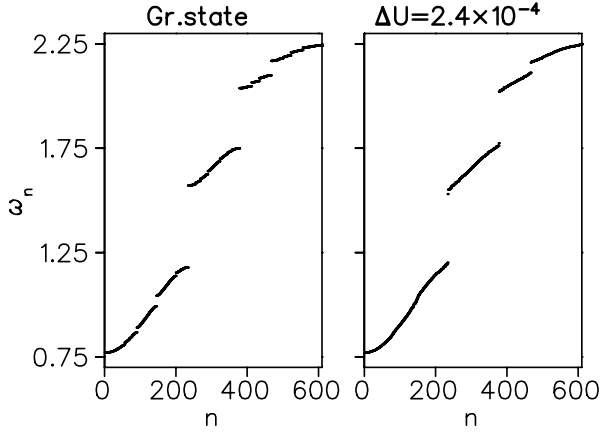


Figure 3: Comparison of the ground state (left box) and excited state (right box) frequency spectra.

degenerate CES, originating from different bricks permutation is combinatorically large, therefore some arbitrary chosen CES has rather *random* sequence of bricks, and is similar to disordered media, e.g. spin glass system.

From the very beginning it is clear, that perfect randomization of bricks order may result in Anderson localization of phonon modes. A new interesting point is, that FK model does *not* require any external disorder: randomization of bricks order in FK chain occurs dynamically. Another interesting feature, which characterizes a degree of chaotization of bricks in CES, is that all the examples of exponential localization are obtained from *single* arbitrary taken CES, without averaging over any ensemble of nearest CES.

A typical quantity used traditionally in studies of localization phenomena (see, in particular [17, 23]) is the participation ratio (PR), defined as

$$R = \frac{1}{s} \left( \sum_{i=1}^s \psi_i^4 \right)^{-1}, \quad (14)$$

where  $\psi$  is a normalized ( $\sum_i \psi_i^2 = 1$ ) vector of the phonon eigenstate. Its value correspond to a chain fraction occupied by the localized state, but whether the state is localized exponentially can be unclear. Typical feature of exponentially localized state is that components outside the center of localization are exponentially small. Meanwhile, this components do not contribute in PR at all.

To our opinion a better characteristics to indicate an *exponential* localization can be the generalized mean geometrical value (MGV), defined as

$$W = s \cdot \left( \prod_{i=1}^s \psi_i^2 \right)^{1/s} = s \exp\left(\frac{1}{s} \sum_{i=1}^s \ln(\psi_i^2)\right), \quad (15)$$

where, a normalization factor  $s$  provides that for extended states  $W$  be order of unity. The reason in favor to  $W$  is that it essentially better probes the exponentially small components (tails) of the phonon eigenstate.

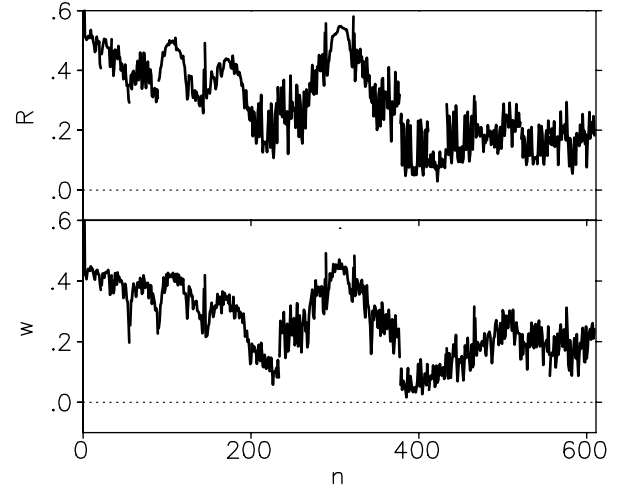


Figure 4: Comparison of PR (upper plot) and MGV (bottom plot) versus phonon mode number  $n$  for the ground state of the chain.

In particular, for a typical exponentially localized state  $|\psi_i| \sim \ell^{-1/2} \exp(-|i - i_0|/\ell)$ ,  $\ell$  is a localization length, and value of the MGV:  $W \sim \ell^{-1} \exp(-s/2\ell)$ , i.e. becomes *exponentially* small. Moreover, one can get from the value of MGV an estimate of the localization length as

$$\ell = -s/2 \ln \left( \frac{W\ell}{s} \right) \simeq -s/2 \ln (-W/2 \ln W). \quad (16)$$

Note an important difference between the estimate of the localization length as inverse participation ratio and our estimate (16). The former estimates a size of the domain where the eigenstate is localized, and is sensitive to particular short range dynamics for formation of the given state. On the contrary, the latter is related to the rate of the exponential falloff at the tails of the eigenstates, and characterizes properties of the disordered media.

Now let us turn to localization properties of phonons in the FK chain. As in the previous section, we concentrate on the numerical study of the chain with the rotation number  $\nu = r/s = 377/610$ , and parameter  $K = 2$ . In Fig.4 we present our comparison of PR and MGV for GS. One can see, that both quantities  $R$  and  $W$  look very similar. Note, that  $W$  for GS is well distinct from zero (dotted line). This means, that all phonon states are *not* exponentially localized; the smallness of  $R$  and  $W$  for some phonon modes means only that this modes are prelocalized only, which agree with earlier studies [23, 25].

In order to see, how the exponential localization manifest itself in PR and MGV, let us address to CES. We start with a typical CES, which is still in energy very close to GS:  $\Delta U = U_{CES} - U_{GS} = 10^{-12}$ . Despite to, that energy splitting is small, this CES belong to 5th band in the energy structure of CES [2], and it is one of  $10^9$  CESs with  $\Delta U \lesssim 10^{-12}$ . In Fig.5 we plot  $R$  and  $W$  for the same CES. It is seen that the behaviour of PR remains in main

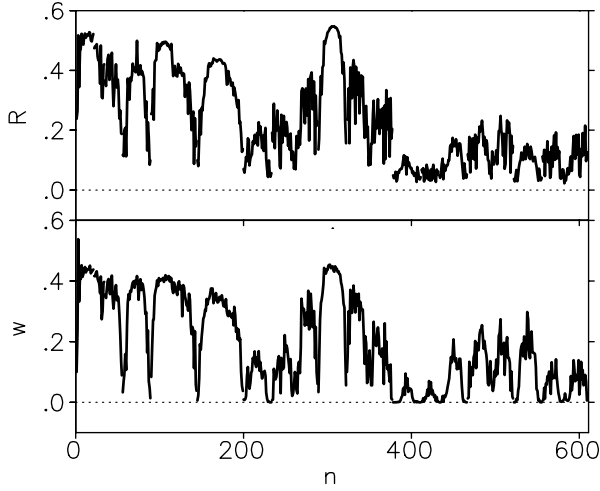


Figure 5: Comparison of PR (upper plot) and MGVS (bottom plot) versus phonon mode number  $n$  for a typical CES with  $\Delta U = 10^{-12}$ .

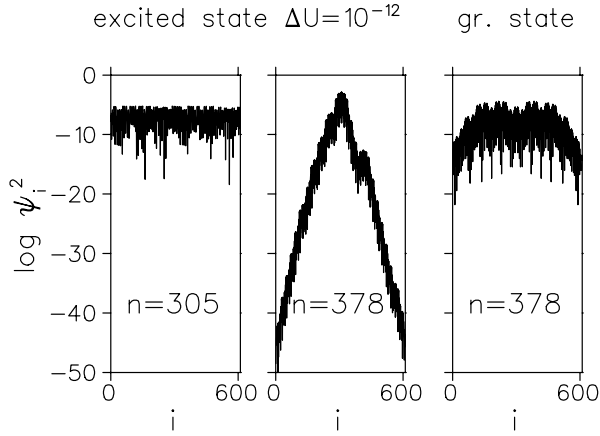


Figure 6: Phonon eigenvector component distribution versus the component number  $i$ . Examples of non-localized and localized modes.

details the same, while in the MGVS plot there are points, where the curve touches a *zero* line. This means, that  $W$  at these points is exponentially small, i.e. there is an *exponential* localization of these modes. Note, that the PR plot at the same points has no clear indications that these modes are localized exponentially.

Some typical examples of nonlocalized mode ( $n = 305$ ) and localized one ( $n = 378$ ) for this CES is presented in Fig.6. We see, that the latter is perfectly localized exponentially, with a localization length  $\ell \sim 12$ , which characterizes a correlation length of disorder. For comparison we show also the same mode in the GS in a “prelocalized” state.

In fact, CES with  $\Delta U = 10^{-12}$  corresponds an early localization of phonon modes: only small fraction of the is exponentially localized, as seen from Fig.5. To get insight, what modes are localized at the first turn, in Fig.7 we plot MGVS as a function of phonon frequency.

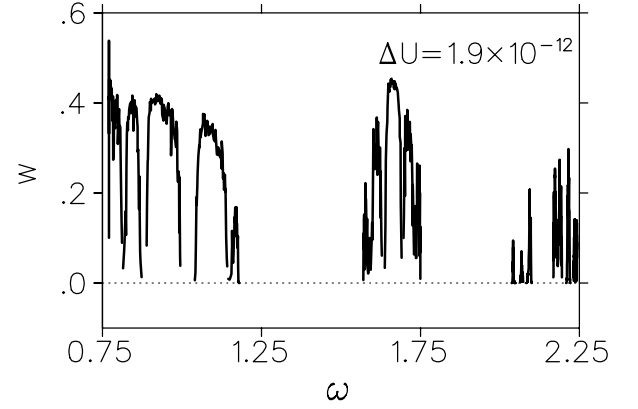


Figure 7: MGVS versus the frequency  $\omega$  of the phonon. It is seen, that phonon are exponentially localized at the edges of frequency bands.

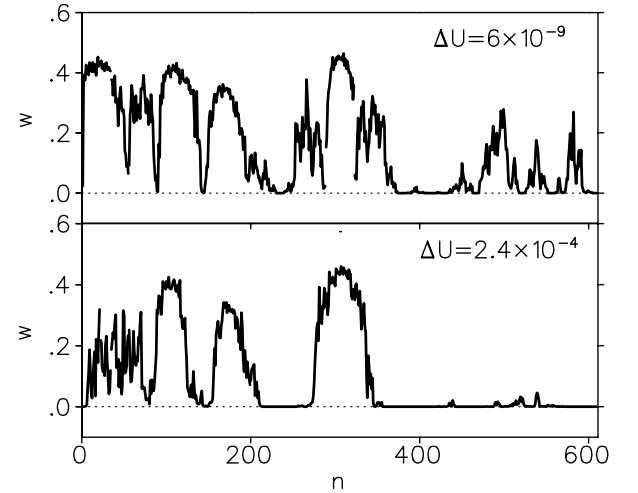


Figure 8: The MGVS versus phonon mode number  $n$  in two CES.

We see, that localized modes are located at edges of the frequency spectrum bands.

Now, how this picture localization looks for CES with higher splitting from GS? We expect [2], that at the splitting  $\Delta U = 10^{-12}$  the largest robust elementary cells[33] are 13 and 21, that take part in mutual permutations only, while larger cells are destroyed. In the range of splitting  $\Delta U = 10^{-9} \div 10^{-8}$  the cell 21 can dissociate [2], that increase a number of smaller cells 13 and decrease a correlation length of the disorder. Next, at splitting  $\Delta U = 10^{-5} \div 10^{-4}$  the cell 13 can dissociate too, and permutations of cells 5 come into play, that decrease a correlation length of the disorder even more. In apparent agreement with our expectations decrease of the disorder scale results in a total exponential localization of substantial fraction of phonon modes, especially in high frequency region, as seen from Fig.8.

In Fig.9 we plot for the same two CES our estimate for localization length (16). Two dotted lines show levels  $\ell = 5$  and 13, which correspond to expected sizes of

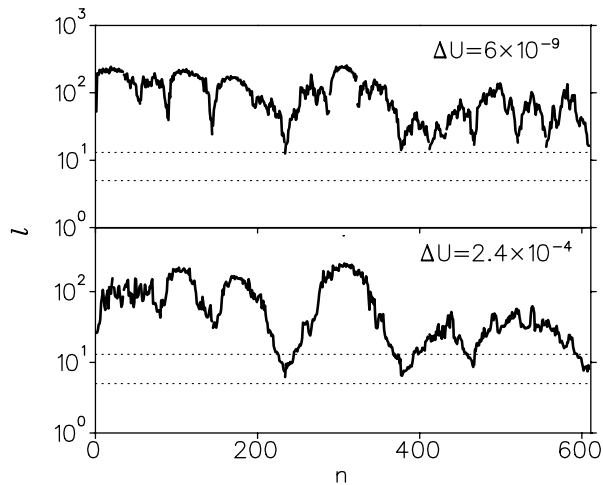


Figure 9: Localization length  $\ell$  versus phonon mode number  $n$ . Horizontal dotted lines correspond to levels  $\ell = 5$  and  $13$ .

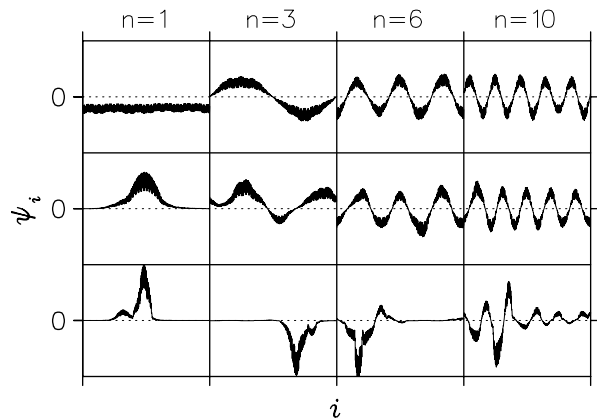


Figure 10: Profiles of phonon eigenstates: eigenvector component distribution versus component number. The upper corresponds to GS, while next two present CES with splitting from GS  $\Delta U = 6 \cdot 10^{-8}$  and  $2.4 \cdot 10^{-4}$  respectively.

largest robust elementary cells, which survive at energy splitting  $\Delta U \sim 10^{-4}$  and  $10^{-8}$ , respectively. We see, that minimal localization length follows the size of maximal robust structure in the chain.

Since the chain is not homogeneous, localization properties are not homogeneous too. As seen from Figs. 8,9, the localization is maximal at the high frequency part of phonon spectrum, while low frequency part seems non-localized. In fact, at least longwave modes of phonons

show a clear tendency to be localized too, see Fig.10.

## V. DISCUSSION AND CONCLUSIONS.

In this paper we have studied properties of phonon modes in the Frenkel-Kontorova chain, taken in the regime of pinned phase. Spatial "brick" structure of the ground state of the chain found in [2], appears to be very useful for understanding the fine structure of the chain phonon spectrum. Actually, similar analysis can also be performed for electronic spectrum, studied in recent work [29].

However, it is obvious, that in a real physical scale the ground state is highly degenerated with a huge number of the static configurational excitations states (CES) of the chain. Actually this means that the true ground state of the chain is practically inaccessible. CES has properties quite different from that of the ground state: their spatial structure is rather chaotic [2]. As a result, they can cause the Anderson-like (exponential) localization of phonon modes similar to that seen in disordered media. This means, that results of previous studies of phonon [17, 18, 23, 25] and electron properties [29] should be revized or extended to more realistic states of the chain.

Configurational excitation states (CES) can provide a possibility to study a gradual transition from the order to disorder. It is important, that the number of CES grows with energy splitting from the GS very fast. Even at very small splitting the number of ECS is huge. It is curious that each particular CES has intrinsic chaos, which reminds the situation in classical spin glass [28], but in contrast to the latter, the chaos in ECS arises dynamically, without any external noise.

The quantization of FK model in small  $\hbar$  limit is in essence the quantization of its phonon modes [24]. Therefore localization of phonon modes means localization of quantum states. Note also, that phonon quantization problem is very close to electron quantization problem [29], where transition from localization to delocalized state is interesting as a insulator-metal transition.

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  - [30] The number of brick species depends on the number properties of the ratio of a mean interparticle distance to a period of the external field, the chain “rotation number”  $\nu$ . The most simple picture quoted here assumes that this rotational number approximates the golden mean value.
  - [31] Note, that at any level of hierarchy a number of glue particles is equal to the total number of bricks.
  - [32] Here two central subbands of widths 21 and 13 are merged due to their overlapping.
  - [33] Let us remind (see previous section), that an elementary cell is a brick with one glue particle added.